Dynamics of vector fields in dimensions 1, 2 and 3

A "quick and dirty" intro

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Objectives:

- What's the problem?
- Can we solve it?
- What kind of dynamics are there?
- How can I make a strange attractor?
- What should we expect?

Outline

Vector fields and flow lines

Dimension 1

Dimension 2

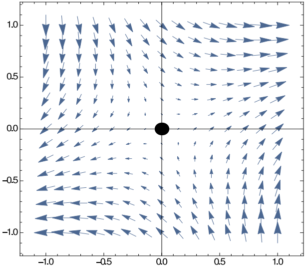
Dimension 3

Vector fields and flow lines

A vector field of $\mathbb{R}^n$ can be represented by a function of the form $f : \mathbb{R}^n \to \mathbb{R}^n$. We shall always assume f to be C^∞.

Example

$f : \mathbb{R}^2 \to \mathbb{R}^2,\ f(x, y) = (x + y, x - y).$



A curve of $\mathbb{R}^n$ is a function of the form $r : \mathbb{R} \to \mathbb{R}^n$ (usually called parametric form of the curve).

Example

The straight line $x = y$ of the plane has the parametric form $r : \mathbb{R} \to \mathbb{R}^2,\ r(t) = (t, t)$.

The circle of the $x - z$ plane of $\mathbb{R}^3$, having the origin as its center and radius 1 can be represented as $r : \mathbb{R} \to \mathbb{R}^3,\ r(t) = (\cos(t), 0, \sin(t))$.

Let f be a vector field of $\mathbb{R}^n$. Is there a curve r of $\mathbb{R}^n$ such that, at every point p of the curve, the vector $f(p)$ is the tangent vector of the curve?

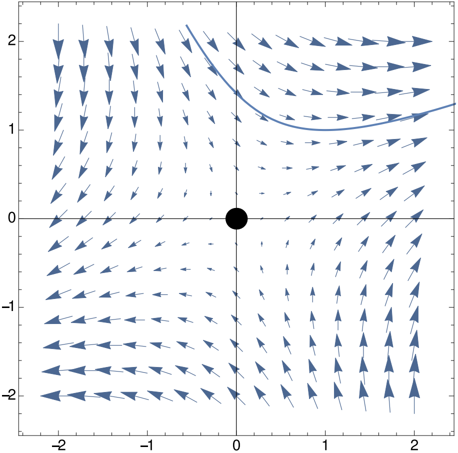
In other words, is it true that $\dot{r}(t) = f(r(t))$, for some curve $r : \mathbb{R} \to \mathbb{R}^n$, passing through the point $r(0) = p \in \mathbb{R}^n$?

Theorem

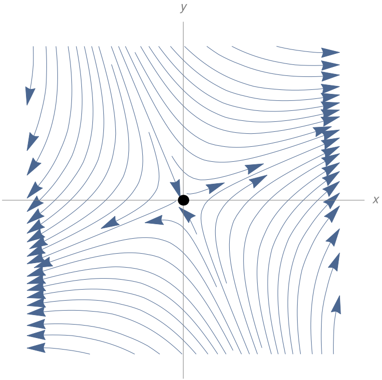
Yes. And this curve is unique. (existence and uniqueness theorem for solutions of ODE's)

Example

For the vector field $f(x, y) = (x + y, x - y)$ and the point $p = (1, 1)$ the curve we are looking for is $r(t) = (\cosh(\sqrt{2}t) + \sqrt{2}\sinh(\sqrt{2}t), \cosh(\sqrt{2}t))$.



If we repeat the procedure to find all the curves passing from all the points of $\mathbb{R}^2$, we construct the "phase space" of our vector field.



Fundamental problem of Dynamical Systems:

For every vector field, draw its phase portrait.

That's the problem. The Existence Theorem assures that it does possess a solution. Yet....

Dimension 1

In dimension 1, vector fields are just good, old functions of the form $f : \mathbb{R} \to \mathbb{R}$. To compute their phase curves (that is, functions of the form $r : \mathbb{R} \to \mathbb{R}$), one should just solve the ode: $\dot{r} = f(r)$. Hopefully, you all know how to do that...

Example

The logistic vector field reads as: $f(x) = x(1-x)$.

To compute its phase curves, one should solve equation: $\dot{r} = r(1-r)$.

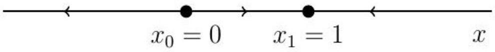
The solution is $r(t) = \frac{x_0 e^t}{1-x_0+x_0 e^t}$, where x_0 is the point for which $r(0) = x_0$.

Observe that, for $x_0 = 0$, $r(t) = 0$ and for $x_0 = 1$, $r(t) = 1$. Dynamical systems theory has another way to draw the phase space of the field.

Solve equation $f(x) = 0$. We find the "fixed points" $x_0 = 0$, $x_1 = 1$.

In the intervals between the fixed points, function $f(x)$ has a constant sign, either positive or negative.

Thus, the phase space is:



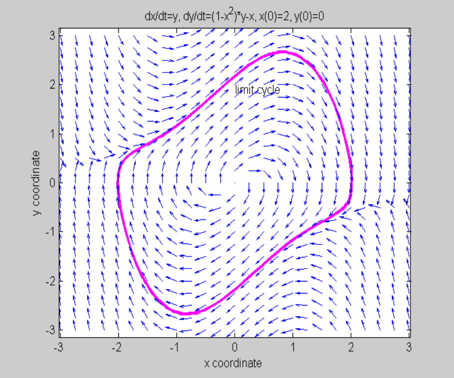
2 fixed points, 1 heteroclinic curve

Try to remember:

1: It is always possible to draw the phase space of a 1–d vector field. It consists just of points and straight segments.

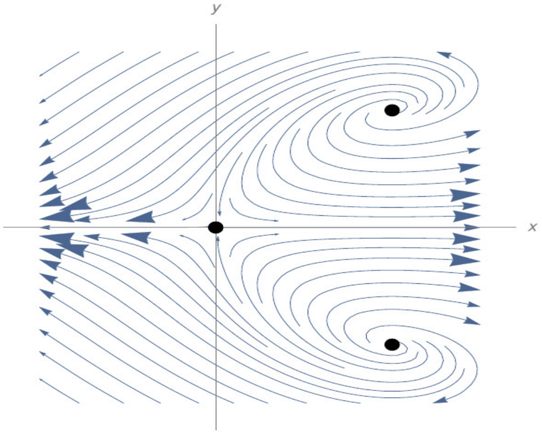
Dimension 2

In 2 dimensions, flow lines have much more space to fill.



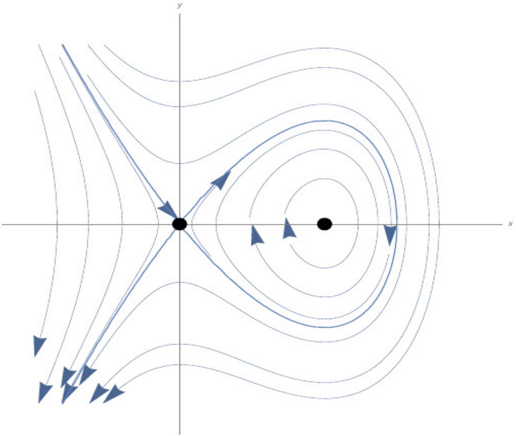
Van der Pol vector field: $f(x, y) = (y, (1 - x^2)y - x)$

1 fixed point, 1 periodic orbit



$$f(x,y) = (x - y^2, -y + xy)$$

3 fixed points, 2 heteroclinic curves



$$f(x, y) = (y, -x^2 + x)$$

2 fixed points, 1 homoclinic curve, an infinity of periodic orbits

- How do we draw these portraits?
  There are various analytical and numerical techniques which, when combined, give us a fairly complete picture.
- What kind of orbits are there?
  Fixed points, periodic orbits, homoclinic and heteroclinic orbits.
- Something more complicated than that?

Well, in dimension 2, the phase space will not be "too" complicated...

Theorem

Poincaré–Bendixson
Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ *and* $p \in \mathbb{R}^2$*. Suppose that the phase curve passing, for* $t = 0$*, through* p *remains for every* $t > 0$ *in a compact subset of* $\mathbb{R}^2$*, which contains finitely many fixed points of* f*. Then:*

- *if the phase curve converges to a set which does not contains fixed points, it converges to a periodic orbit of* f*.*
- *if the phase curve converges to a set that contains both fixed points and orbits which are not fixed points, then these other orbits are curves which converge, both in negative and positive time, to these fixed points.*
- *if the phase curve converges to a set consisting of just fixed points, it converges to a unique fixed point.*

Try to remember:

1: It is always possible to draw the phase space of a 1–d vector field. It consists just of points and straight segments joining them.

2: In dimension 2 the behaviour of a vector field can be much more interesting. There are still many open problems in this area, and sometimes we do have difficulties in sketching an accurate phase space. But, thanks to Poincaré and Bendixson, we can be sure that "too" complicated phase spaces are not to be found.

Dimension 3

To study what kind of dynamics one should expect in dimension 3, we choose the famous Lorenz system:

$$\begin{cases} \dot{x} = \sigma(y - x) \\ \dot{y} = \rho x - y - xz \\ \dot{z} = xy - bz \end{cases} \tag{1}$$

that is, the system of ODE's generated by the vector field

$$f : \mathbb{R}^3 \to \mathbb{R}^3, \; f(x, y, z) = (\sigma(y - x), \rho x - y - xz, xy - bz).$$

Lorenz presented it, at 1963, as a model for weather prediction. And then, he gave a very detailed study of it...

Basic properties

We shall always assume that $\rho,\ \sigma,\ b > 0$.

- Symmetry: Equations are left unchanged under the transformation $(x, y, z) \mapsto (-x, -y, z)$.
- The z-axis is invariant: For $x = y = 0$, $\dot{x} = \dot{y} = 0$.
- Solutions remain bounded as $t \to +\infty$: Let us define the function

$$V : \mathbb{R}^3 \to \mathbb{R},\ V(x, y, z) = \rho x^2 + \sigma y^2 + \sigma(z - 2\rho)^2.$$

Its time derivative equals:

$$\frac{dV}{dt}(x, y, z) = \frac{\partial V}{\partial x}\frac{dx}{dt} + \frac{\partial V}{\partial y}\frac{dy}{dt} + \frac{\partial V}{\partial z}\frac{dz}{dt} =$$
$$= -2\rho\sigma x^2 - 2\sigma y^2 + 4b\rho\sigma z - 2b\sigma z^2.$$

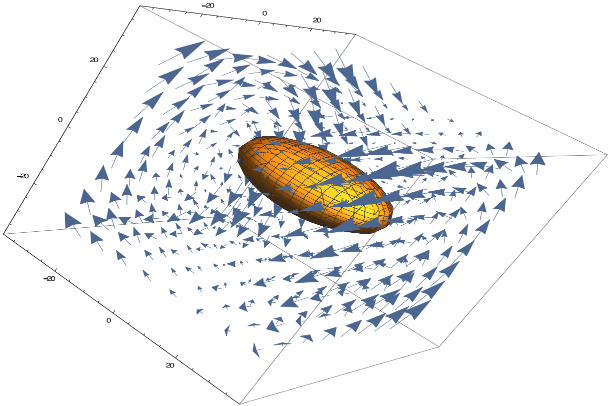
What is the surface:

$$-2\rho\sigma x^2 - 2\sigma y^2 + 4b\rho\sigma z - 2b\sigma z^2 = 0?$$

Equivalently: $\rho x^2 + y^2 + b(z-\rho)^2 = \rho^2$.

Thus, $\dot{V} = 0$ on this ellipsoid, $\dot{V} > 0$ inside this ellipsoid and $\dot{V} < 0$ outside this ellipsoid.

We therefore conclude that, there exists a $c > 0$ such that all orbits crossing ellipsoid $V(x, y, z) = c$ will forever remain into the region bounded by it.



The Lorenz vector field and the bounding ellipsoid. Here, $\sigma = 10,\ b = 8/3, \rho = 2.$

For the Lorenz vector field we can therefore conclude that all “interesting orbits” are contained in the region of $\mathbb{R}^3$ bounded by the ellipsoid found above.

So, what is happening inside this region?

Well, it depends on the parameters (bifurcations occur)....

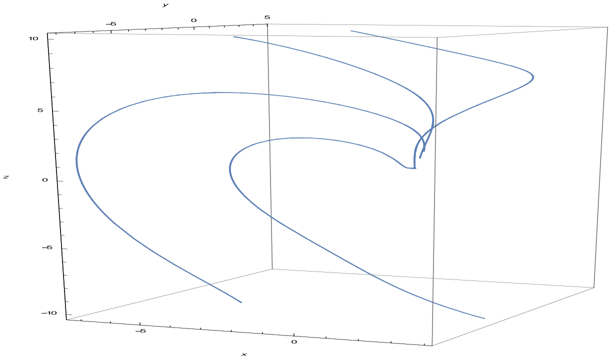
We fix $\sigma = 10,\ b = 8/3$, while $\rho > 0$.

Again, how do we study the system?

The $0 < \rho < 1$ case

Proposition

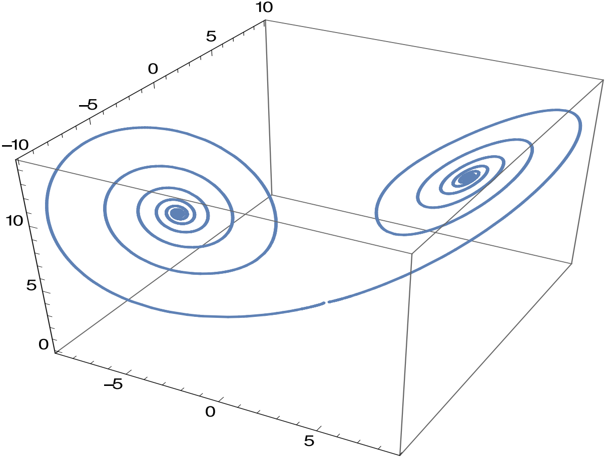
For $0 < \rho < 1$ the origin is the unique fixed point of the Lorenz vector field and it is globally attracting.



Orbits for the Lorenz vector field, $\rho = 1/2$.

The $1 \leq \rho < 13.926$ case

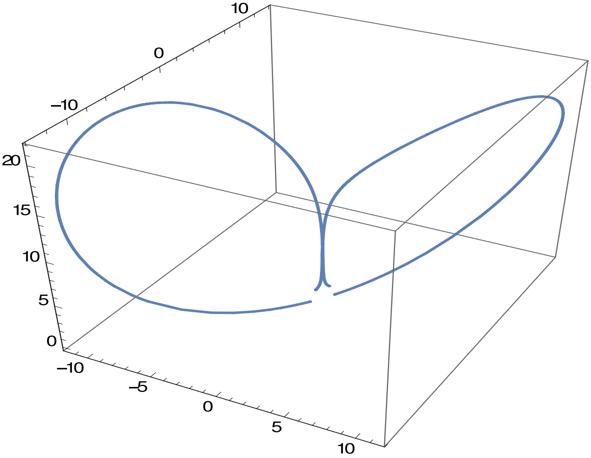
- So, for $0 < \rho < 1$, the origin is a stable fixed point.
- For $\rho > 1$, the origin is a saddle, while two other fixed points have appeared, located at
$$(\pm\sqrt{b\rho - b}, \pm\sqrt{b\rho - b}, \rho - 1).$$
They are both stable.
- Thus, at $\rho = 1$ a pitchfork bifurcation occurs.
- We also see that two heteroclinics appear, connecting the origin with the stable equilibria.



Heteroclinics orbits for the Lorenz vector field, $\rho = 10$, $\sigma = 10$, $b = 8/3$. Initial conditions: $(\pm 0.1, \pm 0.1, 0.1)$.

The $\rho = 13.926$ case

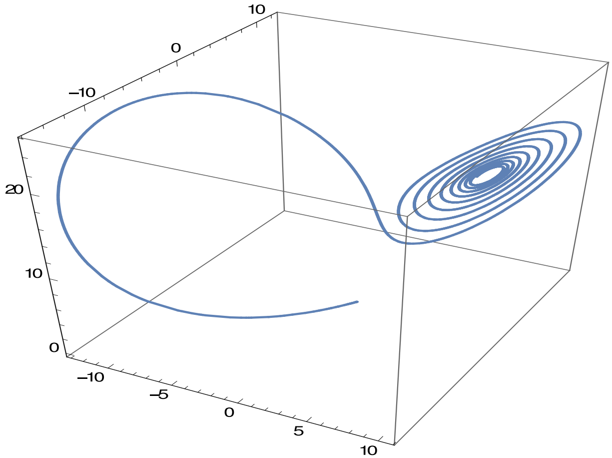
At, approximately, this value of ρ the "homoclinic butterfly" appears. It's unstable, so difficult to draw...



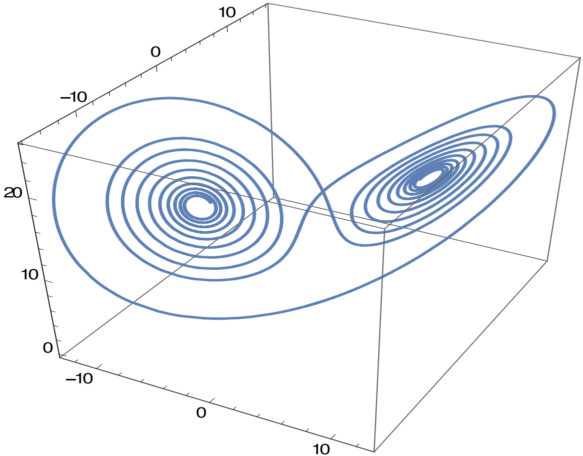
Homoclinic butterfly.

The $\rho > 13.926$ case

- We also have the same number of fixed points.
- There homoclinic butterfly disappears and two heteroclinic orbits emanating from the origin and approaching the stable equilibria appear again.
- Yet, something has changed...

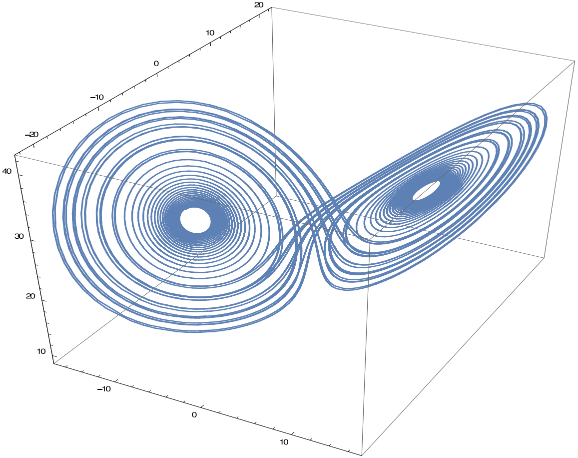


Orbit emanating from the origin approaches one of the non–trivial fixed points, $\rho = 16$, $\sigma = 10$, $b = 8/3$. Initial conditions: $(-0.00974357, -0.017466, 0)$.



The previous orbit and its symmetric one,
$\rho = 16,\ \sigma = 10,\ b = 8/3.$ Initial conditions as above.

- Global bifurcations continue to occur for increasing values of ρ.
- Hopf bifurcations give birth to stable periodic orbits near the non–trivial fixed points.
- For bigger values of ρ the periodic orbits become unstable.
- So, all fixed points are unstable, the periodic orbits born from the Hopf bifurcations are also unstable yet all solutions are still bounded.
- What is happening?



The Lorenz attractor, $\rho = 28,\ \sigma = 10,\ b = 8/3.$

What is it and how do we study it?

Try to remember:

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2: In dimension 2 the behaviour of a vector field can be much more interesting. There are still many open problems in this area, and sometimes we do have difficulties in sketching an accurate phase space. But, thanks to Poincaré and Bendixson, we can be sure that “too” complicated phase spaces are not to be found.

3:In dimension 3 we still observe familiar objects, like fixed points, periodic, homoclinic and heteroclinic orbits, just as in dimension 1 and 2. But it is also possible to come across a new type of behaviour, which cannot be observed in smaller dimensions.

Geometric Lorenz attractors

- What is the Lorenz attractor? First of all, it is an attractor...
- **Definition:** Let X be a vector field of $\mathbb{R}^n$. A compact subset A of $\mathbb{R}^n$ is called an attractor of X if there exists a neighbourhood U of A such that

$$\cap_{t \geq 0} \varphi^t(U) = A.$$

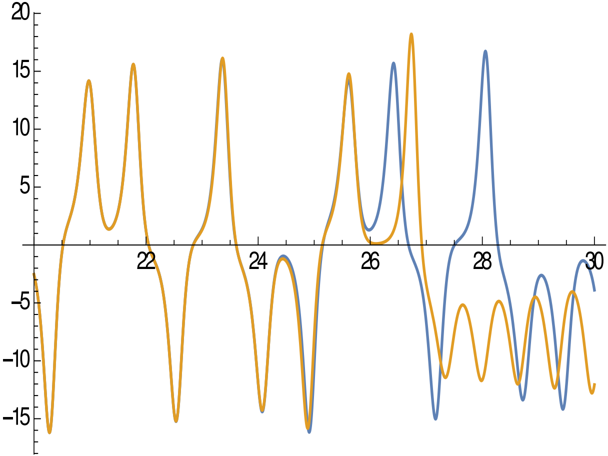
- **Exercise:** Verify that an asymptotically stable fixed point of a vector field is an attractor.

- Lorenz attractor is an attractor: denoting by E the region bounded by the ellipsoid found above, it is

$$\cap_{t \geq 0} \varphi^t(U) = \mathcal{L},$$

where $\mathcal{L}$ is the Lorenz attractor.
- It's not a "simple" attractor (like an attracting fixed point or periodic orbit), since it contains an infinity of orbits.
- We call it chaotic, since it fulfils the definition of chaos (more on this in a while) and it presents sensitive dependence on initial conditions.

Sensitivity on initial conditions



The x projection of two orbits , $\rho = 28,\ \sigma = 10,\ b = 8/3$.
Initial conditions: $(-0.504, -0.86, 0),\ (-0.5, -0.86, 0)$.

So:

- We cannot solve the system of Lorenz.
- Thus, we do not know the parametric form of its phase curves.
- Even if we knew the parametric form of the phase curves, one needs infinite accuracy to locate a specific initial condition.
- And due to sensitive dependence on initial conditions, lack of accuracy leads to long–term unpredictability.

How can one predict future phenomena,
under these conditions?

- Let us recall the words of Poincaré:
  "You are asking me to predict future phenomena. If, quite unluckily, I happened to know the laws of these phenomena, I could achieve this goal only at the price of inextricable computations, and should renounce to answer you; but since I am lucky enough to ignore these laws, I will answer you straight away. And the most astonishing is that my answer will be correct".
  H. Poincaré, Le hasard. Revue du Mois 3, 257276 (1907)

- Following these words, and trying to explain the structure of the Lorenz attractor, Guckenheimer and Williams, and (independently) Afraimovich, Bykov and Shil'nikov, had an idea.
- Forget about equations that can't be solved and initial conditions that are known only to a finite accuracy.
- Focus on what you can describe in a simple and meaningful way.
- So, let us describe what we see in the phase space of the Lorenz system, in a simple and meaningful way.

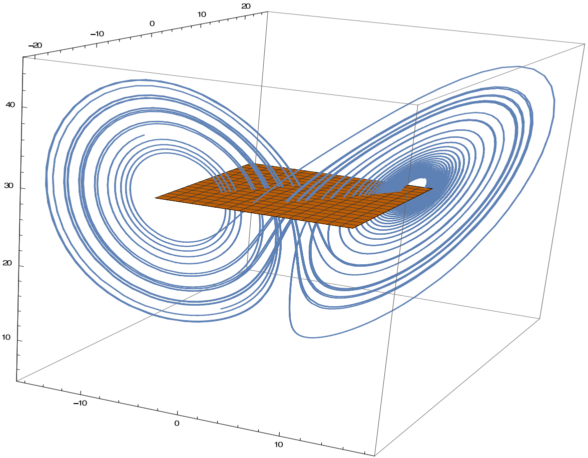
1. Existence of a saddle fixed point

- There exists a fixed point, having one positive eigenvalue $\lambda_1 > 0$ and two negatives $-\lambda_2 < -\lambda_3 < 0$. Moreover, $\lambda_1 > \lambda_3$.
- As a result, there exist two orbits, $\gamma_1(t)$, $\gamma_2(t)$, emanating from the fixed point and "moving away" from it. These two orbits, along with the fixed point, are called the unstable manifold of the point and we denote this manifold as $W^u(0)$.
- There also exists a two dimensional manifold consisting of points the orbits of which tend to the fixed point as $t \to +\infty$. This manifold, denoted by $W^s(0)$, is called the stable manifold of the fixed point.

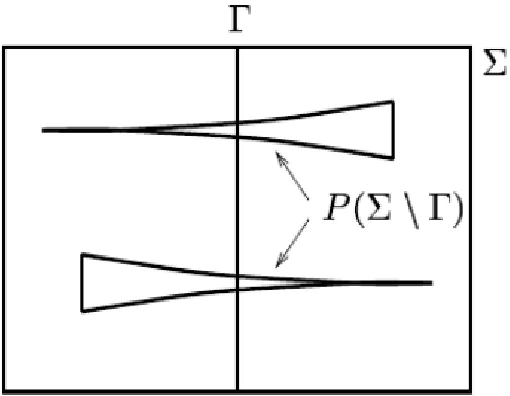
2. Existence of a Poincaré section

- There exists a plane surface Σ of $\mathbb{R}^3$ such that $W^s(0)$ meets it along a curve Γ (we may assume that Γ is a straight line). $W^u(0)$ meets this surface too.
- Every orbit of the vector field with initial condition on one of the components of $\Sigma \setminus \Gamma$ return to Σ after some time $t > 0$.
- Thus, the Poincaré map $P : \Sigma \setminus \Gamma \to \Sigma \setminus \Gamma$ is defined.

Actually, Lorenz had found such a plane section Σ. It is a subset of the $z = \rho - 1$ plane.



3. Assumption: Poincaré map acts as follows

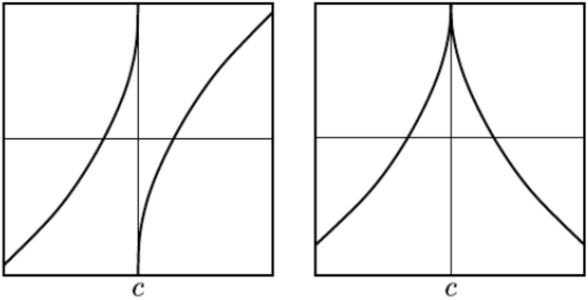


4. Assumption: Existence of an invariant foliation

- Σ can be decomposed in straight line segments, parallel to Γ, invariant under the action of P.
- This means that, if $z_1, z_2 \in \Sigma$ belong to a single line segment, $P(z_1), P(z_2)$ belong to a single (possibly different) line segment as well.
- Furthermore, we demand $P^n(z_1)$ to converge, exponentially fast to $P^n(z_2)$, for $n \to \infty$.

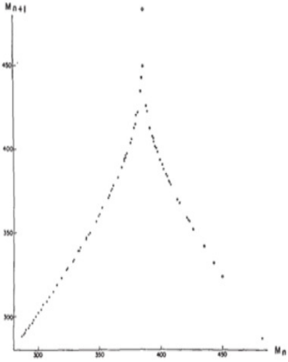
5. Assumption: The one–dimensional mapping

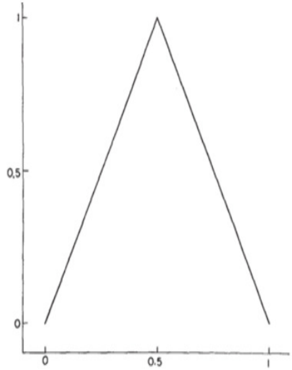
- The previous assumption assures the existence of a mapping $f : [0,1] \to [0,1]$. We demand this mapping to be "expanding enough".



Expanding mappings of the interval.

Actually, Lorenz had computed such an one–dimensional mapping for his system.





Theorem

Every vector field satisfying conditions $1-5$ *above carries an invariant attractor. This attractor contains an infinity of unstable periodic orbits, which are dense in it and at least one orbit which is also dense in the attractor.*

By the way, we just came across the definition of chaos.

Definition: Let K be a compact and invariant subset of the phase space of a vector field. If the set of periodic orbits is dense in K and there also exists a non–periodic orbit which is dense in K, the vector field is said to present chaotic behaviour in K.

Definition: The attractor presented in every system with properties 1–5 above is called geometric Lorenz attractor.

But what about the actual vector field of Lorenz?

Theorem

The Lorenz vector field, for the classical parameter values, carries a geometric Lorenz attractor. (Tucker, 1999)

The proof is another story...

Try to remember:

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2: In dimension 2 the behaviour of a vector field can be much more interesting. There are still many open problems in this area, and sometimes we do have difficulties in sketching an accurate phase space. But, thanks to Poincaré and Bendixson, we can be sure that "too" complicated phase spaces are not to be found.

3: In dimension 3 we still observe familiar objects, like fixed points, periodic, homoclinic and heteroclinic orbits, just as in dimension 1 and 2. But it is also possible to come across a new type of behaviour, which cannot be observed in smaller dimensions.

4: Yes, chaos leads to long–term unpredictability. But we can define it, we can prove that it exists and we can totally study it. It is also quite “simple”: you can produce it with just 5 ingredients.

A glimpse of ergodic theory

Back to Lorenz:

"If a flap of a butterfly's wing can be instrumental in generating a tornado, it can equally well be instrumental in preventing a tornado. More generally, I am proposing that over the years minuscule disturbances neither increase nor decrease the frequency of occurrence of various weather events such as tornados; the most they may do is to modify the sequence in which these events occur."

How do we "measure"?

Definition: Let S be a family of subsets of $\mathbb{R}^n$, closed with respect to the complement and union or intersection of finite number of its members. A function $\mu : S \to [0, +\infty)$ is called a measure on $(\mathbb{R}^n, S)$ if:

- $\mu(\varnothing) = 0$
- $\mu(\cup S_i) = \sum \mu(S_i)$
- $\mu(X) \leq \mu(Y)$ for $X \subseteq Y$. The triple $(\mathbb{R}^n, S, \mu)$ is called a measurable space. If, in addition, $\mu(\mathbb{R}^n) = 1$, measure μ is called a probability measure.

Example: The interval $(0, 1)$, $S =$ the family of its intervals, $\mu =$ the usual "length".

Remark: Obviously, measures are EXTREMELY important in all areas of science.

Definition: If $(\mathbb{R}^n, S, \mu)$ is a space with measure, a flow $\varphi^t : \mathbb{R}^n \to \mathbb{R}^n$ is called measurable, if $\varphi^t(A) \in S$, $\forall A \in S$ and $\forall t \in \mathbb{R}$. We say that the flow preserves the measure, if $\mu(\varphi^t(A)) = \mu(A)$, $\forall A \in S$ and $t \in \mathbb{R}$.

Theorem

Let the flow φ^t preserve a probability measure μ of $\mathbb{R}^n$ and A be a measurable set of $\mathbb{R}^n$. Then, for almost all $x \in A$ there are infinitely many $t \in \mathbb{R}$ such that $\varphi^t(x) \in A$. (Poincaré)

Sinai–Ruelle–Bowen

Definition: An invariant probability measure μ for the flow φ^t is called a SRB–measure if, for every positive Lebesque measure set of points $x \in \mathbb{R}^n$ and every smooth $h : \mathbb{R}^n \to \mathbb{R}$,

$$\lim_{T \to +\infty} \frac{1}{T} \int_0^T h(\varphi^t(x))dt = \int h(x)d\mu.$$

Remark: This actually means that time and "space" averages coincide.

Definition: The support of a measure μ is the set of all points of $\mathbb{R}^n$, every open neighbourhood of which has positive measure.

Theorem

The Lorenz system admits a unique SRB–measure, for which measure the support is exactly the Lorenz attractor.

(Tucker, 1999)

Remark: Time and space averages coincide inside the Lorenz attractor (what a nice property!) and one could try to further study the statistical properties of this object.

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4: Yes, chaos leads to long–term unpredictability. But we can define it, we can prove that it exists and we totally can study it. It is also quite “simple”: you can produce it with just 5 ingredients.

5: Measure theory can also be used to study chaotic attractors in a meaningful way. Actually, there exists a branch of mathematics focusing on this subject: it is called ergodic theory.

“Almost all” dynamical systems

Can we describe what kind of dynamics “most” dynamical systems present?

There is actually a plethora of phenomena that can occur:

- hyperbolic behaviour
- singular hyperbolic behaviour
- homoclinic tangencies
- heterodimensional cycles
- singular cycles
- who knows what else?

But now we are equipped with a vocabulary which permits us to state same conjectures.

Conjecture: The properties below are "prevalent" among dynamical systems on compact manifolds.

- There exist finitely many attractors.
- Each attractor admits a SRB–measure.
- The union of the supports of these measures covers almost all the manifold.

Jacob Palis, 1995.

The butterfly

The butterfly of Lorenz taught us quite a few.

- There exist kinds of complicated behaviour different from what we expected to see before Lorenz.
- Finitely many bifurcations can lead from extremely simple to extremely complicated behaviour.
- Complicated behaviour cannot defy our analytical/numerical tools.
- Topology, geometry, measure theory and a good computer are invaluable to further enrich our understanding.
- There are still a lot to do!

Try to remember:

1: It is always possible to draw the phase space of a 1–d vector field. It consists just of points and straight segments joining them.

2: In dimension 2 the behaviour of a vector field can be much more interesting. There are still many open problems in this area, and sometimes we do have difficulties in sketching an accurate phase space. But, thanks to Poincaré and Bendixson, we can be sure that "too" complicated phase spaces are not to be found.

3: In dimension 3 we still observe familiar objects, like fixed points, periodic, homoclinic and heteroclinic orbits, just as in dimension 1 and 2. But it is also possible to come across a new type of behaviour, which cannot be observed in smaller dimensions.

4: Yes, chaos leads to long–term unpredictability. But we can define it, we can prove that it exists and we totally can study it. It is also quite "simple": you can produce it with just 5 ingredients.

5: Measure theory can also be used to study chaotic attractors in a meaningful way. Actually, there exists a branch of mathematics focusing on this subject: it is called ergodic theory.

6: Don't quit: a scientist is not someone who knows everything, but someone who is willing to look everything up.

For Further Reading I

Clark Robinson
Dynamical Systems
CRC Press, 1998.

Anatole Katok, Boris Hasselblatt
Introduction to the Modern Theory of Dynamical Systems
Cambridge University Press, 1995.

Étienne Ghys
The Lorenz attractor: a paradigm for chaos
Chaos, 1-54, 2013.